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To show that, for every value of n , the relation

$$x^3 + ay^3 + a^2z^3 - 3axyz = v^n$$

has (when a is an integer) an infinite number of integral solutions in x , y , and z .

The function $x^3 + ay^3 + a^2z^3 - 3axyz$ expressed as a circulant is:

$$\begin{vmatrix} x & y\sqrt[3]{a} & z\sqrt[3]{a^2} \\ z\sqrt[3]{a^2} & x & y\sqrt[3]{a} \\ y\sqrt[3]{a} & z\sqrt[3]{a^2} & x \end{vmatrix}$$

We then can assume $(x_1^3 + ay_1^3 + a^2z_1^3 - 3ax_1y_1z_1)^n = x^3 + ay^3 + a^2z^3 - 3axyz$.

Put $n=2$. Then

$$(x_1 + y_1\sqrt[3]{a} + z_1\sqrt[3]{a^2})^2 = x + y\sqrt[3]{a} + z\sqrt[3]{a^2}.$$

From which, by expanding and equating irrationals,

$$x = x_1^2 + 2ay_1z_1$$

$$y = z_1^2a + 2x_1y_1$$

$$z = y_1^2 + 2x_1y_1\sqrt[3]{a}$$

The number of values that can be assigned to x_1 , y_1 , z_1 is infinite; hence the theorem is proved for $n=2$. When $n>2$ it may be proved by proceeding in the same manner.

In Article 9, entitled, "Of the manner of finding Algebraic Functions of all degrees, which when multiplied together may always produce similar functions" in the Additions to Euler's Algebra, Lagrange gives developments which may be considered as applications of the circulant theorem.

Bala, Pa., July 13, 1901.

THE BETWEENNESS ASSUMPTIONS.

By DR. GEORGE BRUCE HALSTED.

In his "Vorlesung ueber Euklidische Geometrie" (Wintersemester 1898-99), Hilbert gave the most remarkable set of axioms ever created for the founding of geometry. These were published in his Festschrift, 1899, treated of in my "Supplementary Report" to the A. A. A. S., (*Science*, Nov. 8, 1901). They have been given in English, though in part erroneously, in D. E. Smith's "The Teaching of Elementary Mathematics," (Macmillan).

The first group of these assumptions Hilbert calls *Axioms of Association*. Of these we need only mention—

I 1. Two distinct points, A , B , always determine a straight a .

I 7. On every straight there are at least two points; in every plane there are at least three non-co-straight points; and in space there are at least four non-co-straight, non-co-planar points.

The second group of assumptions Hilbert calls *Axioms of Arrangement*.

It is these that I call the *Betweenness Assumptions*.

Of them Hilbert says in §3: "The axioms of this group define the idea "between," and make possible on the basis of this idea the *arrangement* of the points on a straight, in a plane and in space.

Convention. The points of a straight stand in certain relations to one another, to describe which especially the word "*between*" serves us.

II 1. If A , B , C are points of a straight, and B lies between A and C , then B also lies between C and A .

II 2. If A and C are two points of a straight, then there is always at least one point B , which lies between A and C , and at least one point, such that C lies between A and D .

II 3. Of any three points of a straight there is always one and only one, which lies between the other two.

II 4. Any four points A , B , C , D of a straight can always be so arranged that B lies between A and C and also between A and D , and furthermore C lies between A and D and also between B and D .

DEFINITION. The system of two points A and B , which lie upon a straight a , we call a *sect*, and designate it with AB or BA .

The points between A and B are said to be points of the sect AB or also situated *within* the sect AB ; all remaining points of the straight a are said to be situated *without* the sect AB . The points A , B are called *endpoints* of the sect AB .

II 5. Let A , B , C be three points not co-straight and a a straight in the plane ABC striking none of the points A , B , C : if then the straight a goes through a point within the sect AB , it must always go either through a point of the sect BC or through a point of the sect AC ."

A year ago, while preparing my Report, I discussed with my pupil, R. L. Moore, the interpretation of the term "angeordnet" in II 4, and whether II 4 might not be demonstrated from the other assumptions.

Finally I wrote to Hilbert asking if he recognized the desirability of any change, as I wished to use his axioms as basis for a text-book on geometry.

His answer bears date April 2, 1902, and was received April 14, 1902. It begins: "Ueber Ihre Idee aus meinen Grundlagen eine Schul-Geometrie zu machen, bin ich sehr erfrunt." Finally, at the very end of the letter, in a post-script, Hilbert says; "Instead of II 4 I believe it suffices simply to say: If B lies between A and C and C between A and D , then lies also B between A and D ; and then to prove my old II 4 as theorem."

I read this to Mr. Moore, and suggested his filling in the proof. Early next morning he announced to me that he had demonstrated Hilbert's new axiom, thus totally eliminating II 4, and reducing the Betweenness Assumptions from five to four. Mr. Moore has no intimation that any one has ever tried to prove these theorems. He started simply from the fact that two weeks ago Hilbert thought an assumption necessary of which now the demonstration, as I have written it out from his oral communication, seems of most unexpected simplicity and elegance.

THEOREM I. If B is between A and C , and C is between A and D , then C is between B and D .

PROOF. Let A, B, C, D be on a . Through C take a straight c other than a . On c take a point E other than C . On the straight BE , between B and E take F .

Thus between B and F is no point of c . Now between A and F there can be no point of c , else c would (by II 5) have a point between A and B , since by the construction of F , c cannot have point between B and F . Thus C would be between A and B , contrary to our hypothesis that B is between A and C .

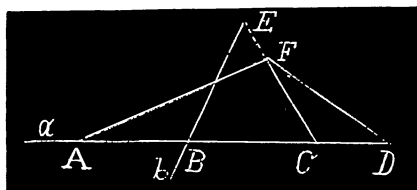
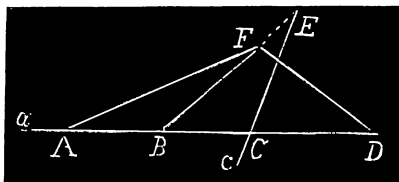
Thus, since c cannot have a point between A and F , it must (by II 5) have a point between F and D . So we have the three non-co-straight points F, B, D , and c with a point between F and D , and, by construction, none between F and B . Therefore it must (by II 5) have a point between B and D . So C is between B and D .

THEOREM II. [Hilbert's new axiom; II 4']. If B is between A and C , and C is between A and D , then B is between A and D .

PROOF. Let A, B, C, D be on a . Through B take a straight b other than a . On b take a point E other than B . On the straight CE , between C and E take F . Thus between C and F is no point of b . Then since by hypothesis B is between A and C , therefore b must (by II 5) have a point between A and F . Thus we have the three non-co-straight points A, F, D , and b with a point between A and F . Therefore b must (by II 5) have a point between A and D or between F and D . But it cannot have a point between F and D , for then it must (by II 5) have a point either between F and C , contrary to our construction, or else between C and D , contrary to Theorem I, by which C is between B and D . Therefore it has a point between A and D . So B is between A and D .

THEOREM III. [Hilbert's old axiom II 4]. Any four points of a straight can always be so lettered $ABCD$, that B is between A and C and also between A and D , and furthermore C is between A and D and also between B and D .

PROOF. We may (by II 3) letter three of our points B, C, D , with C be-



tween B and D . Now as regards B and D , and our fourth point A , either A is between B and D , or B is between A and D , or D is between A and B .

If B is between A and D , we have fulfilled the hypothesis of Theorems I and II. If D is between A and B , then interchanging the lettering for B and D , that is, calling B , D , and D , B , we have again fulfilled the hypothesis of Theorems I and II.

There only remains to consider the case where A is between B and D . If now C is between D and A we have fulfilled the hypothesis of Theorems I and II, by calling D , A , and C , B , and A , C , and B , D . If however A were between C and D we would have fulfilled the hypothesis of Theorems I and II by writing for A , B , for D , A , and for B , D .

We have only left one case to consider, that where D is between A and C . This case is impossible. Suppose $ABCD$ on a . Through C take a straight c other than a . On c take a point E other than C . On the straight DE between D and E take F . Thus between D and F is no point of c .

Then since by hypothesis C is between B and D , therefore c must (by II 5) have a point between B and F . Therefore we have the three non-co-straight points B , F , A , and c with a point between B and F . Therefore c has (by II 5) a point between B and A or a point between F and A .

But it cannot have a point between F and A , else it would (by II 5) have a point between F and D , contrary to our construction, or else between D and A , giving C between D and A , contrary to our hypothesis D between A and C . So C would be between B and A , and D between A and C , and therefore (by Theorem II) D between A and B , contrary to our hypothesis A between B and D .

Thus there is always such a lettering that B is between A and C , and C between A and D , whence (by Theorem I) C is between B and D , and (by Theorem II) B is between A and D .

Austin, Texas, April 17, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A bought a horse, which he sold to B at a loss of $m=6\%$; B sold the horse to C at a loss of $n=6\%$; and C sold the horse to D at a gain of $p=12\frac{1}{2}\%$. How much did A lose, if C gained $G=\$26.79$?